Chapter 7: Laplace Transform

General idea: Laplace transform will change a differential equation to an algebraic one. Solve the new equation → transform back

7.1. Definition and Introduction

Def: Given $f(t)$, the Laplace transform of f is denoted by $\mathcal{L}(f) = F$ and defined by

$$F(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{weight}} f(t)\,dt \,, \quad \text{for } s > 0$$

Remark: The L transform is linear:

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$$

Ex: $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\, 1\, dt = \lim_{a\to\infty} \int_0^a e^{-st}\,dt$$

$$= \lim_{a\to\infty} \left(\frac{e^{-st}}{-s} \Big|_0^a \right) = \lim_{a\to\infty} \left(\frac{e^{-as}}{-s} - \frac{1}{-s} \right)$$

$$= \frac{1}{s} - \underbrace{\lim_{a\to\infty} \frac{e^{-as}}{s}}_{\to 0} = \frac{1}{s}.$$

In summary $f(t) = 1$, $\mathcal{L}(f) = F(s) = \frac{1}{s}$

Ex: $f(t) = e^{at}$, a a fixed constant > 0

(limit)

$$s < a, \quad F(s) = \int_0^\infty e^{-st} e^{at}\, dt$$
$$= \int_0^\infty e^{(a-s)t}\, dt = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$$
$$= \text{undefined}$$

$$s > a, \quad F(s) = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty = -\frac{e^{(a-s)0}}{a-s} = \frac{1}{s-a}$$

In summary, $f(t) = e^{at}$, $\mathcal{L}(f) = F(s) = \frac{1}{s-a}$

Laplace Transform of Basic / Important functions:

f	$\xrightarrow{\text{Laplace}}$	F
$f(t) = 1$		$F(s) = \frac{1}{s}$
$f(t) = e^{at}$		$F(s) = \frac{1}{s-a}$
$f(t) = t^n$		$F(s) = \frac{n!}{s^{n+1}}$
$f(t) = \cos(at)$		$F(s) = \frac{s}{s^2+a^2}$
$f(t) = \sin(at)$	$\xleftarrow{\text{Inverse Laplace}}$	$F(s) = \frac{a}{s^2+a^2}$

⊕ $\mathcal{L}(f(t))$ and $\mathcal{L}(f'(t))$

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\, dt,$$

By integration by parts

$$\left(\int u\,dv = uv - \int v\,du\right)$$

$dv = f'\,dt$, $v = f$

$u = e^{-st}$, $du = (-s)\,e^{-st}$

$$= \underbrace{f(t)\,e^{-st}\Big|_0^{\infty}} - \underbrace{\int f\,(-s)\,e^{-st}\,dt}_{(-s)\,\mathcal{L}(f)}$$

$$= -f(0) + s\,\mathcal{L}(f)$$

In summary: $\mathcal{L}(f'(t)) = s\,\mathcal{L}(f(t)) - f(0)$.

Similarly, $\mathcal{L}(f''(t)) = s^2\,\mathcal{L}(f(t)) - s\,f(0) - f'(0)$

In general,

$$\mathcal{L}(f^{(n)}(t)) = s^n\,\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - s^0 f^{(n-1)}(0).$$

$\Rightarrow$ How the Laplace transform gets rid of derivatives.

7.2. The Inverse Laplace Transform

= Reversing the Laplace transform

Def. Given a function $F(s)$ the inverse L transform $\mathcal{L}^{-1}(F) = f$ if and only if $\mathcal{L}(f) = F$

Ex: $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

$\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, $s > a$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$$

<u>Partial Fraction</u> Method: Inverse L transform of Rational Functions

$$\frac{P(s)}{(s-r_1)\cdots\cdots(s-r_n)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$$

$$\mathcal{L}^{-1}\left(\frac{A_1}{s-r_1}\right) = A_1 e^{r_1 t}, \quad \cdots\cdots$$

$$\frac{P(s)}{(s-r)^m} = \frac{A_1}{s-r} + \cdots + \frac{A_m}{(s-r)^m}$$

$\mathcal{L}^{-1}\left(\frac{A_m}{(s-r)^m}\right)$ is related to $\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

$$\frac{P(s)}{((s-\alpha_1)^2+\beta_1^2)\cdots\cdots((s-\alpha_\ell)^2+\beta_\ell^2)} = \frac{A_1 s + B_1}{(s-\alpha_1)^2+\beta_1^2} + \cdots$$

$$+ \cdots\cdots + \frac{A_\ell s + B_\ell}{(s-\alpha_\ell)^2+\beta_\ell^2}.$$

Ex: Solve $\overset{\text{ODE}}{\boxed{y' - y = 1}}$ with $y(0) = 0$.

Using the Laplace transform $Y = \mathcal{L}(y)$

$$\mathcal{L}(y' - y) = \mathcal{L}(1)$$

$\mathcal{L}(y') - Y = \frac{1}{s}$

Recalling the Laplace transform on derivatives of functions:

$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$

$= sY.$

$\boxed{sY - Y = \frac{1}{s}}$ not involving any derivatives

$(s-1)Y = \frac{1}{s} \Rightarrow Y = \frac{1}{s(s-1)}$

$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right)$

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{-1}{s} + \frac{1}{s-1}$

Solve for A, B: $1 = A(s-1) + Bs$

$s = 0,\ 1 = A(-1) \Rightarrow A = -1$

$s = 1,\ 1 = B \Rightarrow B = 1.$

$y = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right) = \mathcal{L}^{-1}\left(-\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$

$\boxed{y = -1 + e^t}$

Summary: ODE $\xrightarrow{\text{Laplace}}$ equation w/o derivatives $\xrightarrow{\text{solve it}}$ Y = rational function $\rightarrow$ $y = \mathcal{L}^{-1}(Y)$ (partial fraction)

Ex: $y'' - y = 1$ with $y(0) = 0, \; y'(0) = 1$

$Y = \mathcal{L}(y)$

Sol: $\mathcal{L}(y'' - y) = \mathcal{L}(1)$

$s^2 Y - s\,y(0) - y'(0) - Y = \frac{1}{s}$

$s^2 Y - 1 - Y = \frac{1}{s}$

$Y(s^2 - 1) = \frac{s+1}{s} \Rightarrow Y = \frac{s+1}{s(s-1)(s+1)} = \frac{1}{s(s-1)}$

$Y = \frac{1}{s(s-1)}$

$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s}\right)$

$= e^t - 1.$